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POWERS AND ALMOST POWERS

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Powers and almost powers*)

by

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ABSTRACT

An integer of the form ax^m, where $a \in \mathbb{N}$ is given and $x,m \in \mathbb{N}$ with $m \geq 2$ are arbitrary, is called an almost power (with exponent m). Two distinct almost powers with equal exponents not equal to 2 are not contained in a relatively short interval. For three distinct almost squares the same is true. Any number, greater than 1, of distinct positive integers in any very short interval do not have a power as their product.

KEY WORDS & PHRASES: *(almost) powers, short intervals*

*) This report will be submitted for publication elsewhere.

§1. ALMOST POWERS IN SHORT INTERVALS

It is easy to see that two distinct powers with the same exponent, i.e. integers of the form x^m, y^m where $x,y \in \mathbb{N}$ and $m \in \mathbb{N}$ with $m \geq 2$, cannot have a small difference. We now consider also almost powers, i.e. integers of the form ax^m, where a is some given positive integer. The first theorem expresses that two distinct almost powers with the same exponent, but not almost squares, cannot be close together.

<u>THEOREM 1.1</u>. *For every* a, b, M, ε *with* $a,b,M \in \mathbb{N}$, $M \geq 3$ *and* $\varepsilon > 0$ *there exists a positive number* $C = C(a,b,M,\varepsilon)$ *such that for every interval* $[N,N+K]$ *which contains two distinct integers of the form* ax^m, by^m *with* $x,y,m \in \mathbb{N}$ $(x,y) \neq (1,1)$ *and* $m \geq M$ *one has*

$$K > CN^{1-2/M-\varepsilon}.$$

Observe that the number C does not depend on m, but only on M. We also remark that the dependence of the positive numbers $C(a,b,M,\varepsilon)$ on a and b is such that they are bounded (from below) be some positive number $C'(p_1,\ldots$ $\ldots,p_t,M,\varepsilon)$ if a and b are from the set $\{p_1^{v_1}\ldots p_t^{v_t} \mid v_i \in \mathbb{Z},\ v_i \geq 0\}$ of integers composed of primes $p_1,\ldots,p_t$. In particular, an interval of the form $[N,N+c(\varepsilon)N^{1/3-\varepsilon}]$ never contains two distinct integers of the form $2^\alpha x^m$, $3^\beta y^m$ with $x,y,m \in \mathbb{N}$, $\alpha,\beta \in \mathbb{Z}$ and $m \geq 3$, $\alpha \geq 0$, $\beta \geq 0$. See also Theorem 2 in [1].

Two distinct almost squares can be as close as one can (reasonably) wish, as is shown in Theorem 1.3.a. Our second Theorem 1.2 shows that three distinct almost squares cannot be close together.

<u>THEOREM 1.2</u>. *For every* a, b, c, ε *with* $a,b,c \in \mathbb{N}$ *and* $\varepsilon > 0$ *there exists a positive number* $C = C(a,b,c,\varepsilon)$ *such that for every interval* $[N,N+K]$ *which contains three distinct integers of the form* ax^2, by^2, cz^2 *with* $x,y,z \in \mathbb{N}$ *one has*

$$K > CN^{1/4-\varepsilon}.$$

Theorem 1.3 shows that the lower bounds for the lengths of the intervals in Theorems 1.1 and 1.2 cannot be much improved.

THEOREM 1.3.

a) *For every* $M \in \mathbb{N}$ *with* $M \geq 2$ *there exist infinitely many* $N \in \mathbb{N}$ *and a positive number* $C(M)$ *(with* $C(2) = 1$*) such that* $[N, N+C(M)N^{1-2/M}]$ *contains two distinct integers of the form* x^M, $2y^M$ *with* $x, y \in \mathbb{N}$.

b) *There exist infinitely many* $N \in \mathbb{N}$ *and a constant* C *such that* $[N, N+CN^{1/4}]$ *contains three distinct integers of the form* $2x^2$, $3y^2$, $6z^2$ *with* $x, y, z \in \mathbb{N}$.

Apart from the remark following Theorem 1.1, nothing is known about the way the numbers C of the Theorems 1.1 and 1.2 depend on a, b $(,c)$ and ε. To prove certain theorems in §2 we need effective versions of Theorems 1.1 and 1.2, which we state in Theorems 1.4 and 1.5. Also compare Theorem 1.4 with Theorem 1 in [1].

THEOREM 1.4. *Suppose* $[N, N+K]$, *where* $3 \leq K \leq N^{\sigma}$ *with* $0 < \sigma < 1$, *contains two distinct integers of the form* ax^m, by^m, *where* $a, b, x, y, m \in \mathbb{N}$ *with* $m \geq 3$ *and* $m = 3$ *if* $x = y = 1$, *while* a *and* b *are m-free. If* $b/a = 1$ *we put* $C(b/a) = 1$. *If* $b/a \neq 1$, *let* $b/a = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ *with* $\alpha_i \in \mathbb{Z} - \{0\}$, p_i *prime* $(1 \leq i \leq t)$ *and* $p_1 < \ldots < p_t$. *Put* $P = \max\{p_t, 3\}$, $H = \max\{a, b, 3\}$ *and put* $C(b/a) = \min\{C_0 \log H \log\log H, (t+1)^{C_0 t}(\log P)^t \log\log P\}$, *where* C_0 *is a certain absolute (large) constant. Then*

(1) $$m(\log m)^{-1} \leq (1-\sigma)^{-1} C(b/a),$$

(2) $$H^{Cm^3} \log K \log\log K > \log N,$$

where C *is some absolute (large) constant.*

One can also prove inequalities of the form $C(H,m)\log K > \log N$, but in our applications these would not be improvements of (2).

THEOREM 1.5. *Suppose* $[N, N+K]$, *where* $K \geq 3$, *contains three distinct integers of the form* ax^2, by^2, cz^2 *with* $a, b, c, x, y, z \in \mathbb{N}$. *Put* $A = \max\{a, b, c, 3\}$. *Then*

$$CA^2(\log A)^3(A\log A + \log K)(\log A + \log\log K) > \log N,$$

where C *is some (large) absolute constant.*

§2. NEIGHBOURING INTEGERS WHICH HAVE A POWER AS THEIR PRODUCT

In 1975 ERDÖS and SELFRIDGE [2] proved the following elegant assertion, which had been a conjecture for more than 150 years.

PROPOSITION. (Erdös, Selfridge). *The product of two or more consecutive positive integers is never a power.*

The following Theorem 2.1 implies that the product of two or more distinct integers is never a power if the integers are sufficiently large and have an average distance not much greater than 1 (i.e. if the integers $n_1,\ldots,n_f$ satisfy $n_0<n_1<\ldots n_f$ and $(n_f-n_1)(f-1)^{-1} < 1+c(\log f)^{-1}$ for certain positive constants n_0 and c), generalising the proposition on consecutive integers ($(n_f-n_1)(f-1)^{-1} = 1$). Note, however, that in the latter case the constant n_0 may be taken as $n_0 = 0$, by the proposition. See also Theorem 2.2.b for integers for which the average distance is bounded.

THEOREM 2.1. *Let* $n_1,\ldots,n_f$ *be distinct* $(f \geq 2)$ *integers in* $[N,N+K]$ *with* $N \geq N_0$ *and* $f \geq K-cK(\log K)^{-1}$, *where* N_0 *and* c *are certain positive constants. Then*

$$\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m \quad \textit{for any } m \in \mathbb{N} \textit{ with } m \geq 2 \textit{ and any } (m_1,\ldots,m_f) \textit{ with } m_i \in \mathbb{N} \textit{ and } \gcd(m_i,m) = 1 \textit{ for } i = 1,\ldots,f.$$

Theorems 2.2 and 2.3 show that the product of two or more distinct positive integers is never a power if the integers are neighbouring, i.e. contained in a relatively short interval. In the proofs we use Theorems 1.4 and 1.5.

THEOREM 2.2.

a) *For every* $(f,m) \in \mathbb{N}^2$ *with* $f \geq 2$ *and* $m \geq 2$ *there exist positive numbers* c *and* $\varepsilon(f,m)$ *(with* $\lim_{f\to\infty} \varepsilon(f,m) = \lim_{m\to\infty} \varepsilon(f,m) = 0$*) such that if* $n_1,\ldots,n_f$ *are distinct integers in* $[N,N+c(\log N)^{\varepsilon(f,m)}]$ *then* $\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ *for any* $(m_1,\ldots,m_f)$ *with* $m_i \in \mathbb{N}$ *and* $\gcd(m_i,m) = 1$ *for* $i = 1,\ldots,f$.

b) *For every* $(\delta,m) \in (0,1] \times \mathbb{N}$ *with* $m \geq 2$ *there exist positive numbers* c *and* $\varepsilon(\delta,m)$ *(with* $\lim_{\delta\to 0} \varepsilon(\delta,m) = \lim_{m\to\infty} \varepsilon(\delta,m) = 0$*) such that if* $n_1,\ldots,n_f$

are distinct integers in $[N,N+K]$ *with* $K \leq c(\log N)^{\varepsilon(\delta,m)}$ *and* $f \geq \delta K+1$ *then* $\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ *for any* $(m_1,\ldots,m_f)$ *with* $m_i \in \mathbb{N}$ *and* $\gcd(m_i,m) = 1$ *for* $i = 1,\ldots,f$.

While Theorem 2.2 deals with neighbouring integers whose number or average distance is bounded, the first assertion of Theorem 2.3 implies that the product of any number (greater than 1) of any distinct integers in an interval of the form $[N,N+\log\log N]$ is not a square, a cube, ..., an m-th power when N is large enough. However, it does not include the cases where m is 'large', e.g. $m = [\log\log N]$. The second assertion of Theorem 2.3 shows that the product of two or more distinct integers from a still shorter interval is never a power.

<u>THEOREM 2.3</u>.

a) *Let* $m \in \mathbb{N}$ *with* $m \geq 2$. *Let* $n_1,\ldots,n_f$ *be distinct* $(f \geq 2)$ *integers in an interval* $[N,N+cm^{-8}(\log\log N)^2(\log\log\log N)^{-1}]$, *where* c *is some positive absolute constant. Then* $\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ *for any* $(m_1,\ldots,m_f)$ *with* $m_i \in \mathbb{N}$ *and* $\gcd(m_i,m) = 1$ *for* $i = 1,\ldots,f$.

b) *Let* $n_1,\ldots,n_f$ *be distinct* $(f \geq 2)$ *integers in* $[N,N+c\log\log\log N]$, *where* c *is some positive absolute constant. Then* $\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ *for any* $m \in \mathbb{N}$ *with* $m \geq 2$ *and any* $(m_1,\ldots,m_f)$ *with* $m_i \in \mathbb{N}$ *and* $\gcd(m_i,m) = 1$ *for* $i = 1,\ldots,f$. *In particular,* $\prod_{i=1}^{f} n_i$ *is not a power.*

It is, probably, difficult to relax the conditions $\gcd(m_i,m) = 1$ in Theorems 2.2 and 2.3 (it is impossible when m is a prime, of course) in view of the following. Two powers a^p, b^q satisfy $n_1^q n_2^p \in \mathbb{N}^m$ with $m = pq$; it is not known that two distinct powers with distinct exponents cannot be close together (apart from finitely many exceptions such as 3^2 and 2^3). The only known general result is due to TIJDEMAN [3]: apart from finitely many exceptions, two powers are not consecutive integers.

Finally we consider the problem of the existence of (as short as possible) intervals which do contain two or more distinct integers having a power as their product.

<u>THEOREM 2.4</u>. *For* $N \geq 3$ *we write* $K(N) = \exp(12(\log N\log\log N)^{1/2})$. *For every* $m \in \mathbb{N}$ *with* $m \geq 2$ *there exists an infinite set* $N_m \subset \mathbb{N}$ *such that for every*

$N \in N_m$ *the interval* $[N, N+K(N)]$ *contains two or more distinct integers, say* $n_1, \ldots, n_f$, *and integers* $m_1, \ldots, m_f$ *from* $\{1, \ldots, m-1\}$, *with* $\prod_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$.

COROLLARY. *There exist infinitely many positive integers* N *such that* $[N, N+K(N)]$ *contains two or more distinct integers having a square as their product.*

For $m \in \mathbb{N}$ with $m \geq 3$ we cannot find intervals shorter than $[N, N+c_m N^{1-2/m}]$ which contain two or more distinct integers having an m-th power as their product.

§3. PROOFS OF THE THEOREMS IN §1

PROOF OF THEOREM 1.1. Suppose $N \leq ax^m < by^m \leq N+K \leq 2N$. First we consider the trivial case where $b/a = c^m/d^m$ for certain $c,d \in \mathbb{N}$ with $\gcd(c,d) = 1$. Then d^m divides a and we obtain

$$Kd^m \geq d^m(by^m - ax^m) = a((cy)^m - (dx)^m) \geq a((dx+1)^m - (dx)^m) > am(dx)^{m-1}$$

$$= a^{1/m} d^{m-1} (ax^m)^{1-1/m} \geq d^m N^{1-1/m} \geq d^m N^{1-1/M},$$

hence $K > N^{1-1/M}$. Now we assume that the real algebraic number $(b/a)^{1/m}$ is irrational. By Roth's theorem on rational approximation of algebraic irrationals (see Proposition 1, page 11) we have $|(b/a)^{1/m} y - x| > cy^{-(1+\varepsilon)}$ for every $\varepsilon > 0$, where c is some positive number depending only on $\theta = (b/a)^{1/m}$ and ε. Observe that $by^m - ax^m = a(\theta y - x)((\theta y)^{m-1} + (\theta y)^{m-2} x + \ldots + x^{m-1})$. Put $z = \min\{x,y\}$. Since $x,y \gg N^{1/m}$ we have $z \gg N^{1/m}$. Also $y^m \leq N+K \leq 2N$. Hence

$$K \geq by^m - ax^m \geq acy^{-(1+\varepsilon)} m \cdot \min\{b/a, 1\} \cdot z^{m-1} \gg N^{(m-2-\varepsilon)/m}$$

$$> N^{1-2/m-\varepsilon}.$$

Hence $K > c_1(a,b,m,\varepsilon) N^{1-2/m-\varepsilon}$ for some positive number c_1 depending only on a, b, m and ε. We also have

$$by^m - ax^m = ax^m(by^m/ax^m - 1) > ax^m(m\log(y/x) + \log(b/a)) \neq 0.$$

By a theorem on linear forms in logarithms (see Proposition 3, page 12) we have, writing $Z = \max\{x,y\}$ (≥ 2),

$$(3) \qquad m\log(y/x) + \log(b/a) > \exp(-C\log Z\log m)$$

for some positive number C depending only on a and b. Since $Z^m \leq N+K \leq 2N \leq N^2$ we obtain

$$K \geq by^m - ax^m > ax^m\exp(-2Cm^{-1}\log m \log N) \geq N^{1-2Cm^{-1}\log m}.$$

Hence for $m > m_0(a,b,M)$, a certain number depending only on a, b and M, we have $K > N^{1-2/M}$. Put $c_2 = c_2(a,b,M,\varepsilon) = \min_{M\leq m\leq m_0} c_1(a,b,m,\varepsilon)$. Then we have $K > c_2N^{1-2/m-\varepsilon} \geq c_2N^{1-2/M-\varepsilon}$ for $M \leq m \leq m_0$ and $K > N^{1-2/M}$ for $m > m_0$. This implies the assertion of Theorem 1.1. □

REMARK. The number C in (3) depends only on the distinct prime divisors of a and b. Since c_1 depends, in fact, only on the m-free part of a and b, it follows that the final constant $C = \min\{1,c_2\}$ in Theorem 1.1 depends only on the distinct prime divisors of a and b and on ε and M.

PROOF OF THEOREM 1.2. Suppose $N \leq ax^2 < by^2 < cz^2 \leq N+K \leq 2N$. If one of $\sqrt{b/a}$, $\sqrt{c/a}$, $\sqrt{c/b}$ is rational, then we have, trivially, $K > N^{1/2}$. Hence we may assume that these algebraic numbers are all irrational, or, equivalently, that 1, $\sqrt{c/a}$ and $\sqrt{c/b}$ are linearly independent over the rationals. By Schmidt's theorem on simultaneous rational approximation of algebraic irrationalities (Proposition 1) we have, therefore,

$$|z\sqrt{c/b} - y|\cdot|z\sqrt{c/a} - x| > c_0z^{-(1+\varepsilon)},$$

for some positive number c_0 depending only on ε, $\sqrt{c/b}$ and $\sqrt{c/a}$. Hence

$$K^2 \geq (cz^2-by^2)(cz^2-ax^2)$$

$$= ab(z\sqrt{c/b} - y)(z\sqrt{c/a} - x)(z\sqrt{c/b} + y)(z\sqrt{c/a} + x)$$

$$\gg z^{-(1+\varepsilon)}(\min x,y,z)^2 \gg N^{(1-\varepsilon)/2}.$$

This proves Theorem 1.2. □

PROOF OF THEOREM 1.3.

a) The assertion for $M = 2$ follows from the (well-known) existence of infinitely many $x,y \in \mathbb{N}$ with $x^2 - 2y^2 = 1$. Assume $M \geq 3$. By Dirichlet's theorem on rational approximation of irrationals (Proposition 2) there exist infinitely many $x,y \in \mathbb{N}$ with $|y2^{1/M}-x| < y^{-1}$ and $x,y > M^{1/2}$. For these integers x, y we have

$$|2y^M-x^M| = |y2^{1/M}-x|((y2^{1/M})^{M-1} + \ldots + x^{M-1}) < y^{-1}\cdot 2M(\max\{x,y\})^{M-1}.$$

Put $N = \min\{x^M,2y^M\}$ then it follows that x^M and $2y^M$ belong to the interval $[N,N+K]$ for some $K \ll MN^{1-2/M}$.

b) By Dirichlet's theorem on simultaneous rational approximation (Proposition 2) there exist infinitely many $x,y,z \in \mathbb{N}$ with

$$|z\sqrt{3} - x| < z^{-1/2} \quad \text{and} \quad |z\sqrt{2} - y| < z^{-1/2}.$$

In particular $y \ll x \ll z \ll x \ll y$. It follows that

$$|6z^2-2x^2| = 2|z\sqrt{3} - x|(z\sqrt{3} + x) \ll z^{-1/2}\max\{x,z\} \ll (\max\{x,y,z\})^{1/2},$$

$$|6z^2-3y^2| = 3|z\sqrt{2} - y|(z\sqrt{2} + y) \ll z^{-1/2}\max\{y,z\} \ll (\max\{x,y,z\})^{1/2}.$$

Let $N = \min\{6z^2,2x^2,3y^2\}$, then it follows that $6z^2,2x^2,3y^2 \in [N,N+K]$ for some $K \ll N^{1/4}$. □

PROOF OF THEOREM 1.4. Suppose that $N \leq ax^m < by^m \leq N+K$, where $3 \leq K \leq N^\sigma$, $0 < \sigma < 1$. If $b/a = 1$ then

$$N^\sigma \geq K \geq a(y^m-x^m) \geq a((x+1)^m-x^m) > amx^{m-1} > (ax^m)^{1-1/m} \geq N^{1-1/m},$$

hence $m(\log m)^{-1} < m \leq (1-\sigma)^{-1} = C(b/a)\cdot(1-\sigma)^{-1}$. We assume $b/a \neq 1$ now. We may also assume that $(x,y) \neq (1,1)$, since otherwise $m = 3$ and inequality (1) of Theorem 1.4 trivially holds since $C(b/a)$ exceeds 3, while inequality (2) trivially holds, if $x = y = 1$, since $H \geq a \geq N$, then. Observe that

$$N^{\sigma} \ge K \ge by^m - ax^m > ax^m(by^m/(ax^m)-1) > ax^m(m\log(y/x)+\log(b/a)) \ge N\cdot\Lambda$$

By Proposition 3 we have, with $Z = \max\{x,y\}$ (≥ 2),

$$\begin{cases} \Lambda = m\log(y/x) + \log(b/a) > \exp(-C\log H\log\log H\log Z\log m), \\ \Lambda = m\log(y/x) + \sum_{i=1}^{t} \alpha_i \log p_i > \exp(-(t+1)^{Ct}(\log P)^t\log\log P\log Z\log m). \end{cases}$$

Since $Z^m \le N+K < 2N \le N^2$ it follows that

$$\begin{cases} N^{\sigma-1} \ge \Lambda \ge \exp(-2C\log H\log\log H\cdot m^{-1}\log m\cdot\log N), \\ N^{\sigma-1} \ge \Lambda \ge \exp(-2(t+1)^{Ct}(\log P)^t\log\log P\cdot m^{-1}\log m\cdot\log N) \end{cases}$$

which implies that (1) holds

To prove (2) we use the following result, proved by EVERSE [4], following a method of Baker.

Let $x,y \in \mathbb{Z}$ satisfy $x^m - dy^m = k$, where $d \in \mathbb{Z}$, $m \in \mathbb{N}$, $k \in \mathbb{Z}$, with $m \ge 3$ and $k \ne 0$. Put $k^* = \max\{|k|,3\}$. Then

$$\max\{|x|,|y|\} < \exp((m|d|)^{Cm^2}\log k^*\log\log k^*),$$

where C is some large absolute constant.

We have $b^{m-1}(by^m - ax^m) = (by)^m - ab^{m-1}x^m = k$, where $b^{m-1} \le k \le b^{m-1}K$. Observe that $d = ab^{m-1} \le H^m$ and $k^* = \max\{3,k\} \le b^{m-1}K < H^mK$. It follows that $N \le by^m < \exp(H^{Cm^3}\log K\log\log K)$ for some large constant C. □

To prove Theorem 1.5 we first prove

<u>THEOREM 1.6</u>. *Let* a, b, c, d *be square free positive integers with* $a \ne b$ *and* $c \ne d$ *and let* e *and* f *be integers. If* $af = ce$ *then we also assume that* abcd *is not a square. Then for every solution* $(x,y,z) \in \mathbb{N}^3$ *of the pair of equations*

$$(4) \qquad \begin{cases} ax^2 - by^2 = e \\ cx^2 - dz^2 = f \end{cases}$$

one has

$$\max\{x,y,z\} < \exp(C\alpha^2(\log\alpha)^3\gamma\log\gamma),$$

where $\alpha = \max\{a,b,c,d\}$, $\beta = \max\{|e|,|f|,3\}$, $\gamma = \max\{\alpha\log\alpha, \log\beta\}$ *and* C *is some large absolute constant.*

In the proof of Theorem 1.6 we use results of Nagell and Schur on equations of the form $X^2 - DY^2 = 1$ (see Proposition 4) and a lower bound for linear forms in logarithms of algebraic numbers (Proposition 3). Our method of proof is analogous to the proof in [5], where it is shown that the only solutions of (4) in the case $(a,b,c,d,e,f) = (3,1,8,1,2,7)$ are those corresponding to $x = 1$ and $x = 11$.

PROOF OF THEOREM 1.6. We may assume that $\gcd(a,b) = \gcd(c,d) = 1$. By (4) we have

$$\begin{cases}(ax)^2 - aby^2 = ae\\ (cx)^2 - cdz^2 = cf\end{cases}$$

Since $aecf \neq 0$ and ab and cd are not squares we conclude from Proposition 4 that

$$(5)\qquad \begin{cases}ax + y\sqrt{ab} = (s+t\sqrt{ab})\varepsilon_{ab}^{n}\\ cx + z\sqrt{cd} = (u+v\sqrt{cd})\varepsilon_{cd}^{m}\end{cases}$$

for certain $s,t,u,v,m,n \in \mathbb{Z}$ with $|s|,|t| \le \varepsilon_{ab}\sqrt{a|e|}$ and $|u|,|v| \le \varepsilon_{cd}\sqrt{c|f|}$. Combining (5) with the conjugate equations we obtain, since $\bar{\varepsilon}_{ab}^{n} = \varepsilon_{ab}^{-n}$ and $\bar{\varepsilon}_{cd}^{m} = \varepsilon_{cd}^{-m}$,

$$(6)\qquad \begin{cases}2ax = (s+t\sqrt{ab})\varepsilon_{ab}^{n} + (s-t\sqrt{ab})\varepsilon_{ab}^{-n}\\ 2cx = (u+v\sqrt{cd})\varepsilon_{cd}^{m} + (u-v\sqrt{cd})\varepsilon_{cd}^{-m}\end{cases}$$

In view of the symmetry in (6) we may assume that $m \ge 0$ and $n \ge 0$. From (6) it follows that

$$(7)\qquad A\varepsilon_{ab}^{n} - B\varepsilon_{cd}^{m} = C,$$

where $A = c(s+t\sqrt{ab})$, $B = a(u+v\sqrt{cd})$, $C = a(u-v\sqrt{cd})\varepsilon_{cd}^{-m} - c(s-t)\sqrt{ab})\varepsilon_{ab}^{-n}$. The minimum polynomial of A is $T^2 - 2csT + c^2ae$. Hence $H(A) = \max\{1, 2c|s|, c^2a|e|\} \le \alpha^{3\alpha}\cdot\beta$ since, by Proposition 4, $\varepsilon_{ab} < (ab)^{\sqrt{ab}} \le \alpha^{2\alpha}$. Analogously, $H(B) \le \alpha^{3\alpha}\cdot\beta$. We also have $H(\varepsilon_D) = 2x_D < 2\varepsilon_D$ and $\varepsilon_D \ge 1 + \sqrt{D} > 2$. Observe that $A > 0$ and $B > 0$. Also note that $B^{-1} = (u-v\sqrt{cd})(acf)^{-1}$, hence $B^{-1} \le \max\{|u|,|v|\}2\sqrt{cd}(ac|f|)^{-1} < \alpha^{3\alpha}$. Also $A^{-1} < \alpha^{3\alpha}$. We also have $|C| \le a|u-v\sqrt{cd}| + c|s-t\sqrt{ab}| \le 2\alpha^{3\alpha}\beta$. Observe that $C \ne 0$: if $C = 0$ then $A\varepsilon_{ab}^{n} = B\varepsilon_{cd}^{n}$ hence $c^2ae = A\bar{A} = B\bar{B} = a^2cf$, so $ec = af$. From (4) we infer $bcy^2 = adz^2$, contradicting that $abcd$ is not a square. Since $C \ne 0$ we may assume, by the symmetry in (7), that $C > 0$.

We now show that $N := \max\{m,n,4\}$ satisfies

$$N(\log N)^{-1} \le C_0\alpha(\log\alpha)^2(\alpha\log\alpha + \log\beta) \tag{8}$$

for some constant C_0.

CASE 1.

$$C > (B\varepsilon_{cd}^{m})^{1/2}$$

In this case we have $m \le \log(C^2B^{-1})/\log\varepsilon_{cd} \ll \alpha\log\alpha + \log\beta$. Since

$$A\varepsilon_{ab}^{n} = C + B\varepsilon_{cd}^{m} < C + C^2 \le 2\max\{1,C^2\} = C^*$$

we obtain

$$n \le \log(C^*A^{-1})/\log\varepsilon_{ab} \ll \alpha\log\alpha + \log\beta.$$

Therefore (8) holds in this case.

CASE 2.

$$(0 <)C \le (B\varepsilon_{cd}^{m})^{1/2}.$$

It follows from (7) that

$$0 < \log(A/B) + n\log\varepsilon_{ab} - m\log\varepsilon_{cd} < C/B\varepsilon_{cd}^{m} < (B\varepsilon_{cd}^{m})^{-1/2} =: \lambda^{-1/2}.$$

Observe that we also have the upper bound $\max\{(\mu/2)^{-1},(\mu/2)^{-1/2}\}$, where $\mu = A\varepsilon_{ab}^{n}$, since $\mu = C + B\varepsilon_{cd}^{m} < \lambda^{1/2} + \lambda \le 2\max\{\lambda,\lambda^{1/2}\}$. We use this upper bound if $n \ge m$ and the bound $\lambda^{-1/2}$ if $n < m$. We may assume now that $4 \le n < m$. From Proposition 3 it follows that

$$\log(A/B) + n\log\varepsilon_{ab} - m\log\varepsilon_{cd}$$

$$> \exp(-C_1\log(\alpha^{3\alpha}\beta)\log H(\varepsilon_{cd})\log(\alpha^{2\alpha})\log\alpha\log m).$$

As we saw, we also have the upper bound $(B\varepsilon_{cd}^{m})^{-1/2} < B^{-1/2}(H(\varepsilon_{cd})/2)^{-m/2} < \exp(3\alpha\log\alpha - 1/2\cdot m\log(H(\varepsilon_{cd})/2))$. Combining the bounds we infer

$$N(\log N)^{-1} = m(\log m)^{-1} << (\alpha\log\alpha + \log\beta)\cdot\alpha\cdot(\log\alpha)^{2}$$

Therefore (8) also holds in this case.

From (6) it follows that $x < 2\max\{|s|,|t|\}\sqrt{ab}\varepsilon_{ab}^{n} < \alpha^{4n\alpha}\beta^{1/2}$. By (8) we have $n << \alpha(\log\alpha)^{2}\gamma\log\gamma$. Hence $x < \exp(C\alpha^{2}(\log\alpha)^{3}\gamma\log\gamma)$ for some constant C. In view of (4) this inequality also holds for y and z for some C. □

PROOF OF THEOREM 1.5. We have $ax^2 - by^2 = e$ and $ax^2 - cz^2 = f$ for some $e,f \in \mathbb{Z}$ with $|e|,|f| \le K$. We may assume that a, b, c are square free and also that they are pairwise distinct (since otherwise $K > N^{1/2}$ and the inequality of Theorem 1.5 holds). Therefore we can apply Theorem 1.6. We obtain

$$x < \exp(CA^{2}(\log A)^{3}(A\log A + \log K)(\log A + \log\log K))$$

for some constant C. Since $x \ge (N/A)^{1/2}$ the inequality of Theorem 1.5 follows. □

Finally we state the results which we used in this §.

PROPOSITION 1. (ROTH, SCHMIDT [6]). *Suppose* $\alpha_1,\ldots,\alpha_n$ *are real algebraic numbers such that* $1,\alpha_1,\ldots,\alpha_n$ *are linearly independent over the rationals and suppose* $\delta > 0$. *Then there exists a positive number* c, *depending only on*

$\alpha_1,\ldots,\alpha_n$ *and* δ, *such that for every* $p_1,\ldots,p_n \in \mathbb{Z}$ *and every* $q \in \mathbb{N}$ *one has*

$$|\alpha_1 q-p_1|\ldots|\alpha_n q-p_n| > cq^{-(1+\delta)}.$$

PROPOSITION 2. (DIRICHLET, [6]). *Suppose that at least one of the real numbers* $\alpha_1,\ldots,\alpha_n$ *is irrational. Then there exist infinitely many distinct n-tuples* $(p_1/q,\ldots,p_n/q)$ *with* $p_i \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *and* $|\alpha_i q-p_i| < q^{-1/n}$ *for* $i = 1,\ldots,n$.

PROPOSITION 3. (BAKER, [7]). *Let* $\alpha_1,\ldots,\alpha_n$ *(where* $n \geq 2$*) be positive algebraic numbers of degree one or two with heights at most* $A_1,\ldots,A_n$, *respectively, where* $A_i \geq 2$ *for* $1 \leq i \leq n-1$ *and* $A_n \geq \max\{A_1,\ldots,A_{n-1},3\}$. *Let* $\beta_1,\ldots,\beta_n$ *be rational integers with absolute values at most* B (≥ 2). *Suppose that* $\Lambda := \Sigma_{j=1}^n \beta_j \log\alpha_j$ *is not zero. Then*

$$\Lambda > \exp(-n^{Cn}\log A_1\ldots\log A_n \log\log A_n \log B)$$

where C *is some (large) absolute constant.*

PROPOSITION 4. (SCHUR, NAGELL, [8], [9]). *Let* D *be a positive integer which is not a square and let* $\varepsilon_D = x_D + y_D\sqrt{D}$, *where* (x_D,y_D) *is the solution of* $x^2 - Dy^2 = 1$ *with* $(x,y) \in \mathbb{N}^2$ *and* $x + y\sqrt{D}$ *minimal. Then* $\varepsilon_D < \exp(D^{1/2}\log D)$. *If the equation*

$$x^2 - Dy^2 = e, \tag{9}$$

where $e \in \mathbb{Z}$ *with* $e \neq 0$, *has solutions* $(x,y) \in \mathbb{Z}^2$ *then there exist (finitely many) solutions* $(s,t) \in \mathbb{Z}^2$ *of* (9) *with*

$$|s|,|t| \leq \varepsilon_D|e|^{1/2}$$

such that every solution $(x,y) \in \mathbb{Z}^2$ *of* (9) *has the form*

$$x + y\sqrt{D} = (s + t\sqrt{D})\varepsilon_D^n$$

for some $n \in \mathbb{Z}$ *and some* (s,t).

§4. PROOFS OF THE THEOREMS OF §2

First we prove the following lemma.

<u>LEMMA</u>. *Let* $m,f \in \mathbb{N}$ *with* $m \geq 2$ *and* $f \geq 2$. *Let* $n_1,\ldots,n_f$ *be distinct integers in the interval* $[N,N+K]$, *where* $K \leq N^{1-1/m}$, *with the property that* $\prod_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$ *for some* $(m_1,\ldots,m_f)$ *with* $m_i \in \mathbb{N}$ *and* $\gcd(m_i,m) = 1$ *for* $i = 1,\ldots,f$. *Write* $n_i = a_i x_i^m$ *with* $a_i, x_i \in \mathbb{N}$ *and* a_i *m-free* $(1 \leq i \leq f)$. *Then*

(1) $P(a_i) \leq K$ *for* $i = 1,\ldots,f$,

(2) $a_i \leq \exp((m-1)(f-1)\log K)$ *for* $i = 1,\ldots,f$,

(3) *there exist at least two (three if* $f \geq 3$*)* a_i*'s with* $a_i \leq \exp(3mKf^{-1}\log K)$,

(4) *there exist at least two (three if* $f \geq 3$*)* a_i*'s with* $a_i \leq \exp(CmK^{1/2}(\log K)^{1/2})$, *where* C *is some absolute constant.*

<u>PROOF</u>. First we observe that $a_i \neq a_j$ for $i \neq j$: if $a_i = a_j$, then, assuming $n_i > n_j$, we have $K \geq n_i - n_j = a_j(x_i^m - x_j^m) > ma_j x_j^{m-1} > N^{1-1/m}$, a contradiction. To prove (1), let $1 \leq i \leq f$ and let p be a prime divisor of a_i. Since $\prod_{j=1}^{f} a_j^{m_j} \in \mathbb{N}^m$, the m_j are relatively prime with m and a_i is m-free, there exists a j with $1 \leq j \leq f$, $j \neq i$ and $p | a_j$. Hence $p | a_i - a_j$ and, since $a_i \neq a_j$, $p \leq |a_i - a_j| \leq K$. To prove (2), let $1 \leq i \leq f$. Observe that a_i divides

$$\prod_{\substack{j=1 \\ j\neq i}}^{f} \prod_{p|a_j} p^{v_p(a_i)}$$

which divides

$$\prod_{\substack{j=1 \\ j\neq i}}^{f} \Big(\prod_{\substack{p|a_j \\ p|a_i}} p\Big)^{m-1}$$

which divides

$$\prod_{\substack{j=1 \\ j\neq i}}^{f} (\gcd(a_i,a_j))^{m-1}.$$

Since $a_i \neq a_j$ for $i \neq j$ we have $\gcd(a_i,a_j) \leq |a_i - a_j| \leq K$ and we infer that $a_i \leq K^{(m-1)(f-1)}$. We now prove (3): we have

$$\prod_{i=1}^{f} a_i = \prod_p p^{\sum_{i=1}^{f} v_p(a_i)} = \prod_p p^{\sum_{j=1}^{m-1} \#\{1\le i\le f \mid p^j \text{ divides } a_i\}}$$

$$\le \prod_{p\le K} p^{\sum_{j=1}^{m-1}(1+[K/p^j])} \le \prod_{p\le K} p^{m-1} \cdot \prod_{p\le K} p^{\sum_{j=1}^{\infty}[K/p^j]}$$

$$= (\prod_{p\le K} p)^{m-1} \cdot K! \le K^{\pi(K)(m-1)+K} \le K^{mK}.$$

If $f \ge 3$ we conclude that the number of a_i with $a_i > \exp(mK\log K \cdot (f-2)^{-1})$ is less than $f-2$, hence there are at least three a_i with $a_i \le \exp(3mKf^{-1}\log K)$. If $f = 2$ then, by (2), a_1 and a_2 do not exceed $\exp((m-1)\log K)$. This proves (3).

Finally we prove (4). Suppose $\lambda \in \mathbb{N}$ with $3 \le \lambda \le f$. We have

$$\prod_{i=1}^{\lambda} a_i \le \mathrm{LCM}[a_1,\ldots,a_\lambda] \cdot \prod_{1\le i<j\le\lambda} \gcd(a_i,a_j) \le \prod_{p\le K} p^{m-1} \cdot K^{\binom{\lambda}{2}}$$

$$\le 3^{mK} K^{\binom{\lambda}{2}}.$$

Hence there exist at least three a_i among $a_1,\ldots,a_\lambda$ with

$$a_i \le \exp((\lambda-2)^{-1}(mK\log 3 + \binom{\lambda}{2}\log K)).$$

If $f \ge 2 + [(K/\log K)^{1/2}]$ then we take $\lambda := 2 + [(K/\log K)^{1/2}]$ (≥ 3) and we obtain three a_i with $a_i \le \exp(CmK^{1/2}(\log K)^{1/2})$ for some constant C. If $f \le 1 + [(K/\log K)^{1/2}]$ then, by (2), $a_i \le \exp(mK^{1/2}(\log K)^{1/2})$ for every $1 \le i \le f$. This proves (4). □

REMARK. The combination of (2) and (3) gives that $a_i \le \exp(3mK^{1/2}\log K)$ for at least two (three) a_i, which is only slightly weaker than (4).

PROOF OF THEOREM 2.1. Suppose that $n_1,\ldots,n_f$ are distinct $(f \ge 2)$ integers in $[N,N+K]$, where $N \ge N_0$ and $f \ge K - cK(\log K)^{-1}$, where $c\,(>0)$ is a sufficiently small constant and N_0 is a sufficiently large constant, and suppose that

$\Pi_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$ for some $m \in \mathbb{N}$ with $m \geq 2$ and some $(m_1,\ldots,m_f)$ with $m_i \in \mathbb{N}$ and $\gcd(m_i,m) = 1$ for $i = 1,\ldots,f$. We shall finally obtain a contradiction, but first we show that $K < N^{1/m}$.

If $\delta < 1/2$ and N is sufficiently large and $K \geq N$ then there exist more than $\delta K(\log K)^{-1}$ primes in $[(N+K)/2, N+K] \subset [N,N+K]$, hence if $c \leq \delta$ and N_0 is sufficiently large then one of the n_i is such a prime (which cannot divide any other $n_j \neq n_i$), contradicting $\Pi_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$. Hence $K < N$. If $N^{2/3} \leq K(< N)$ then the number of primes in $[N,N+K]$ is asymptotically $K(\log N)^{-1}$ for $N \to \infty$, by a well-known result of Ingham, hence, provided $c < 2/3$ and N_0 is sufficiently large, there exists a n_i which is a prime (larger than $N > K$, hence not dividing any other $n_j \neq n_i$), contradicting $\Pi_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$. Hence $K < N^{2/3}$. For $K_0 \leq K < N^{2/3}$, where K_0 is some absolute constant, the number of integers v in $[N,N+K]$ with $P(v) > K$ exceeds $K/6$ (see, if necessary, the proof of Theorem 2 in [10]). Hence, provided $K \geq K_1(c)$, one of the n_i is divisible by a prime p exceeding K. Hence $p \nmid n_j$ for every $n_j \neq n_i$ and from $\Pi_{j=1}^{f} n_j^{m_j} \in \mathbb{N}^m$, it follows therefore that $p^m | n_i$. This implies $(K+1)^m \leq p^m \leq n_i \leq N+K$, hence $K < N^{1/m}$. If $K < K_0$, where K_0 is some constant, then, by Theorem 2.3(b), we have $N < N_0(K_0)$, some constant depending only on K_0. Taking N_0 large enough and c sufficiently small (i.e. $c < 1/2$) we may therefore assume that $K_0 \leq K < N^{1/m}$ in the sequel, where K_0 is some suitable constant. We distinguish the cases $m \geq 3$ and $m = 2$ now.

Suppose $m \geq 3$. In [2] it is shown that it follows from $K < N^{1/m}$, $m \geq 3$ and $P(a_i) \leq K$ (which holds in view of (1) of the lemma) that all products $a_i a_j$, where $1 \leq i, j \leq f$, are distinct (in particular the a_i are distinct), and also that, consequently,

$$\sum_{a_i \leq x} 1 \leq \pi(x) + O(x^{3/4}(\log x)^{-3/2}) = x(\log x)^{-1}(1 + O((\log x)^{-1})).$$

We change (if necessary) the indices of the a_i in such a way that $a_1 < \ldots < a_f$. Then it follows that $t = \Sigma_{a_i \leq a_t} 1 \leq a_t(\log a_t)^{-1}(1+O((\log a_t)^{-1}))$, hence $a_t \geq t\log t + t\log\log t + O(t)$, in particular $a_t \geq t\log t$ for $t \geq t_0$. Therefore

$$\prod_{t=1}^{T} a_t \geq \exp\Big(\sum_{t=1}^{T} \log(t\log t) + O(1)\Big) = \exp(T\log T + T\log\log T + O(T)) \tag{10}$$

for $T \geq 2$. For every prime divisor q of $a_1 \ldots a_f$ we choose an $1 \leq i(q) \leq f$

with $v_q(a_{i(q)}) \geq v_q(a_j)$ for $j = 1,\dots,f$. Then

$$\prod_{\substack{i=1\\ i\neq i(q)\,\forall q}}^{f} a_i = \prod_p p^{\sum_{j=1}^{m-1} \#\{1\leq i\leq f \mid i\neq i(q)\forall q \text{ and } p^j \text{ divides } a_i\}}$$

$$\leq \prod_p p^{\sum_{j=1}^{m-1}[K/p^j]} \leq K!$$

Put $f^* = f - \Pi(K)$ (≥ 2). Since there exist at most $\Pi(K)$ distinct q's we obtain

$$\prod_{t=1}^{f^*} a_t \leq \prod_{\substack{i=1\\ i\neq i(q)\,\forall q}}^{f} a_i \leq K! \leq K^K.$$

Combining this with (10) gives $f^* \leq K(1 - \log\log K\,(\log K)^{-1} + O((\log K)^{-1}))$. It follows that $f \leq K - K(\log K)^{-1}\log\log K + O(K(\log K)^{-1})$, a contradiction with $f > K - cK(\log K)^{-1}$ and $K \geq K_0$, since K_0 is sufficiently large. We now consider the case $m = 2$.

As we saw in the proof of the lemma, $\Pi_{i=1}^{f} a_i$ divides $(K!)\,\Pi_{p\leq K}\, p$. Hence

$$(11) \qquad \prod_{i=1}^{f} a_i \text{ divides } K! \prod_{2<p\leq K} p\cdot 2^{v_2(\Pi_{i=1}^f a_i)-v_2(K!)}\cdot 3^{v_3(\Pi_{i=1}^f a_i)-v_3(K!)}.$$

Note that $p|a_i$ if and only if $v_p(n_i)$ is odd. Let $n(p)$ be an integer in $[N,N+K]$ with $v_p(n(p)) \geq v_p(n)$ for every $n \in [N,N+K]$. Then $p^{v_p(n)} \leq K$ for every $n \neq n(p)$ in $[N,N+K]$. Let $j(p)$ be the maximal integer with $p^{2j+1} \leq K$. Then it follows that

$$v_p(\prod_{i=1}^{f} a_i) \leq 1 + \#\{1\leq i\leq f \mid n_i\neq n(p),\ v_p(n_i) \text{ is odd}\}$$

$$\leq 1 + \sum_{j=0}^{j(p)} (N(2j+1) - N(2j+2)),$$

where $N(i) = \#\{n \in \mathbb{N} \mid N\leq n\leq N+K,\ n\neq n(p),\ p^i|n\} = K/p^i + O(1)$. It follows that $v_p(\Pi_{i=1}^f a_i) \leq K/(p+1) + O(\log K)$. Using also that $K! = \exp(K\log K - K + O(\log K))$, $\Pi_{p\leq K}\, p = \exp(K + O(K(\log K))$ and $v_p(K!) = K/(p-1) + O(\log K)$ we obtain from (11)

$$\prod_{i=1}^{f} a_i \le \exp(K\log K - ((2\log 2)/3 + (\log 3)/4)K + O(K(\log K)).$$

Let $1 = q_1 < q_2 < \ldots$ be the increasing sequence of square free integers. Then $q_t \sim \pi^2 t/6$ for $t \to \infty$. Hence $a_t \ge q_t > dt$ for any $d < \pi^2/6$ and $t \ge t_0(d)$. It follows that $\prod_{i=1}^{f} a_i > f!d^f$ for $f \ge f_0(d)$, hence

$$\prod_{i=1}^{f} a_i > \exp(f\log f + (\log d-1)f + O(\log f)).$$

Combining the bounds for $\prod_{i=1}^{f} a_i$ we obtain $f \le K - ((2\log 2)/3 + (\log 3)/4 + \log d - 1)K/\log K + O(K/\log K)$. Since $(2\log 2)/3 + (\log 3)/4 + \log(\pi^2/6) - 1$ is positive we obtain a contradiction with $K \ge K_0$ and $f \ge K - cK/\log K$ if c is sufficiently small. □

REMARKS.

1) By including the primes 5, 7, 11, 13 in the last argument on the square free a_i one can see that the assertion of Theorem 2.1 holds for any $c < 1/2$ (the 1/2 comes from the first argument in the proof), provided N_0 is sufficiently large.
2) It is implicit in the proof that if $n_1,\ldots,n_f$ are distinct $(f \ge 2)$ integers in $[N,N+K]$ with $N \ge N_0$, $K \le N^{1/3}$ and $f \ge K - cK(\log K)^{-1}\log\log K$ for some $c < 1$ and N_0 sufficiently large, then $\prod_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ for any $m \in \mathbb{N}$ with $m \ge 3$ and any $(m_1,\ldots,m_f)$ with $m_i \in \mathbb{N}$ and $\gcd(m_i,m) = 1$ for $i = 1,\ldots,f$.

PROOF OF THEOREM 2.2. First we suppose that $m \ge 3$, $3 \le K \le N^{1-1/m}$ and that $n_1,\ldots,n_f$ are distinct $(f \ge 2)$ integers in $[N,N+K]$ with $\prod_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m$ for some $(m_1,\ldots,m_f)$ with $m_i \in \mathbb{N}$ and $\gcd(m_i,m) = 1$ for $i = 1,\ldots,f$. Writing $n_i = a_i x_i^m$ with a_i m-free, as in the lemma, we obtain, from (2) of the lemma, two distinct integers $a_1x_1^m$ and $a_2x_2^m$ with $a_i \le K^{(m-1)(f-1)} =: H$ in $[N,N+K]$. From inequality (2) of Theorem 1.4 we obtain

$$K^{Cm^4 f} \log K \log\log K > \log N.$$

Hence $K \gg (\log N)^{\varepsilon(f,m)}$ holds for every $\varepsilon(f,m)$ with $0 < \varepsilon(f,m) < (Cm^4 f)^{-1}$.

Now we consider the case $m = 2$. If $f = 2$, then, trivially, $n_i = ax_i^2$ for some $a \in \mathbb{N}$ and $i = 1,2$, hence $K > N^{1/2}$ and therefore $K \gg (\log N)^{\varepsilon(2,2)}$ holds for every positive number $\varepsilon(2,2)$. We assume $f \geq 3$ now. By (2) of the lemma we have $n_i = a_i x_i^2$, with $a_i \leq K^{f-1}$ for $i = 1,\dots,f$. Hence $[N,N+K]$ contains the distinct integers $a_i x_i^2$, $i = 1,2,3$, where $a_i \leq K^{f-1} =: A$. By Theorem 1.5 we infer that $K^{3(f-1)}(f\log K)^5 \gg \log N$. This implies that $K \gg (\log N)^{\varepsilon(f,2)}$ holds for every $\varepsilon(f,2)$ with $0 < \varepsilon(f,2) < (3(f-1))^{-1}$. This proves part (a) of Theorem 2.2.

To prove part (b) we argue similarly, but we use (3) of the lemma instead of (2). We then obtain $a_i \leq K^{3m\delta^{-1}} =: A$ for at least two (three) a_i. It follows that $K \gg (\log N)^{\varepsilon(\delta,m)}$ for every $\varepsilon(\delta,m)$ with $0 < \varepsilon(\delta,m) < \delta/(9m)$.□

<u>PROOF OF THEOREM 2.3.</u> To prove part (a) we argue the same as in the proof of Theorem 2.2, but we use (4) of the lemma instead of (2) or (3). In case $m \geq 3$ and $\prod_{i=1}^f n_i^{m_i} \in \mathbb{N}^m$ we obtain two distinct integers $a_1 x_1^m$ and $a_2 x_2^m$ in $[N,N+K]$ with $a_i \leq \exp(CmK^{1/2}(\log K)^{1/2}) =: H$. By inequality (2) of Theorem 1.4 we obtain

$$\exp(Cm^4(K\log K)^{1/2}))\log K\log\log K > \log N,$$

which implies $K \gg m^{-8}(\log\log N)^2(\log\log\log N)^{-1}$.

If $m = 2$ then the assertion trivially holds for $f = 2$ as we saw in the proof of Theorem 2.2. Assuming $f \geq 3$ we obtain, by (4) of the lemma, three distinct integers $a_i x_i^2$ with $a_i \leq \exp(C(K\log K)^{1/2}) =: A$. From Theorem 1.5 we infer $\exp(C'(K\log K)^{1/2}) > \log N$ for some suitable $C' > 0$. Hence $K \gg (\log\log N)^2(\log\log\log N)^{-1}$.

We now prove part (b). Suppose $n_1,\dots,n_f$ are distinct ($f \geq 2$) in $[N,N+K]$ and $\prod_{i=1}^f n_i^{m_i} \in \mathbb{N}^m$ for some $m \in \mathbb{N}$ with $m \geq 2$ and some $m_i \in \mathbb{N}$ with $\gcd(m_i,m) = 1$ for $i = 1,\dots,f$. We shall prove that $K \gg \log\log\log N$. If $m = 2$ then this follows from part (a). We assume $m \geq 3$, and also that $K \leq N^{1/2}$. We have $n_i = a_i x_i^m$ with a_i m-free and $a_i \leq \exp(Cm(K\log K)^{1/2})$ for at least two a_i, say for a_1 and a_2. We apply inequality (1) of Theorem 1.4, with $\sigma = 1/2$, and obtain $m/\log m \leq 2C(a_1/a_2)$. We show that $C(a_1/a_2) \leq \exp(CK)$ for some constant C. If $a_1/a_2 = 1$ then this holds, since $C(1) = 1$. By (1) of the lemma we have $P(a_i) \leq K$ for $i = 1,2$. Writing $a_1/a_2 = p_1^{\alpha_1}\dots p_t^{\alpha_t}$ with primes p_i and

$\alpha_i \in \mathbb{Z}-\{0\}$, we have $p_1<\ldots<p_t \le K$ and $t \le \Pi(K)$. In view of the definition of $C(a_1/a_2)$ it follows that $C(a_1/a_2) \le (t+1)^{C_0 t}(\log P)^t \log\log P < \exp(CK)$ for some C. Hence $m < \exp(CK)$ for some constant C and, therefore, $a_i <$ $\exp\exp(CK) =: H$ for some C and $i = 1,2$. From (2) of Theorem 1.4 we conclude that $\exp\exp(CK) > \log N$ for some constant C, in other words $K \gg \log\log\log N$.□

PROOF OF THEOREM 2.4. We shall use the following (elementary) fact (see, for example, [11])

(12) There exists an infinite set S of positive integers such that the number $f(N)$ of integers n in $[N,N+K(N)]$ with $P(n) \le P_0(N) =:$ $\exp((\log N\log\log N)^{1/2})$ satisfies $f(N) > (P_0(N))^2$ for every $N \in S$.

Let $m \in \mathbb{N}$ with $m \ge 2$. For every set T which consists of $[f(N)/2]$ distinct integers n in $[N,N+K(N)]$ with $P(n) \le P_0(N)$ we define $\phi(T)$ as the tuple

$$\phi(T) = \Big(\sum_{t\in T} v_p(t) \bmod m\Big)_{p\le P_0(N),\, p \text{ prime}} .$$

The number of distinct sets T is $\binom{f(N)}{[f(N)/2]}$ and the number of distinct tuples $\phi(T)$ is at most $m^{\Pi(P_0(N))}$. Hence there exists a tuple $v = (v_p)_{p\le P_0(N)} = v(N)$ such that the number of distinct sets T with $\phi(T) = v$ is at least $\binom{f(N)}{[f(N)/2]} m^{-\Pi(P_0(N))}$. In view of (12) there exist for every $N \in S$ with $N \ge N_0(m)$ (defining an infinite set N_m) m distinct sets $T_1,\ldots,T_m$ with $\phi(T_i) = v(N)$ for $i = 1,\ldots,m$:

$$\binom{f(N)}{[f(N)/2]} > 2^{f(N)}/f(N) > 2^{P_0(N)^2}/P_0(N)^2 > m^{\Pi(P_0(N))+1}$$

for $N \ge N_0(m)$, $N \in S$.

Hence $\prod_{t\in T_i} t \in \prod_{p\le P_0(N)} p^{v_p}\cdot \mathbb{N}^m$ for $i = 1,\ldots,m$. Let $n_1,\ldots,n_g$ be the distinct integers in $T_1\cup\ldots\cup T_m$ and let m_i be the number of j with $1 \le j \le m$ and $n_i \in T_j$, for $i = 1,\ldots,g$. Then $1 \le m_i \le m$ for $1 \le i \le g$ and $\prod_{i=1}^g n_i^{m_i} =$ $\prod_{i=1}^m \prod_{t\in T_i} t \in \mathbb{N}^m$, while $\sum_{i=1}^g m_i = m[f(N)/2]$. Deleting those n_i from $n_1,\ldots,n_g$ with $m_i = m$ we obtain distinct integers $n_1',\ldots,n_f'$ in $[N,N+K(N)]$ and integers $m_i' \in \{1,\ldots,m-1\}$ for $1 \le i \le f$ with $\prod_{i=1}^f (n_i')^{m_i'} \in \mathbb{N}^m$. Since

$\Sigma_{i=1}^{f} m_i'$ is a nonnegative multiple of m, but not zero (otherwise $T_1 = \dots = T_m$, a contradiction), we have $f \geq 2$. □

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